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ADAPTIVE CONTROL VIA A SIMPLE SWITCHING ALGORITHM*

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Abstract. In this paper we present an adaptive stabilization control for systems with unknown constant parameters and stochastic disturbances, which may be neither open-loop stable nor minimum phase. The ideas come from previous works [J. F. Zhang and H. F. Chen, *Adaptive stabilization under the weakest condition*, Proc. 31st Control and Design Conference, December 14–18, 1992, pp. 3620–3621, and H. F. Chen, *Continuous-Time Stochastic Adaptive Control Stabilizing the System and Minimizing the Quadratic Loss Function*, Tech. Report, Institute of Systems Science, Academia Sinica, Beijing, 1992], but here we not only simplify the construction procedure of an adaptive control but also reduce the computational load significantly, so that the adaptive control in this paper is more practical. Furthermore, parameter estimation is carried out in only a finite time period and, unlike previous work, the parameter estimates are generated by ordinary differential equations rather than stochastic differential equations.

Key words. adaptive control, parameter estimation, switching algorithm, continuous time, stochastic system

AMS subject classifications. 93C40, 93E15, 93E35

1. Introduction.

The switching control strategies of Zhang and Chen [1991] and Chen [1992] show that an alternation of excitation and control regimes can yield stabilizing controls. The idea is that, if a certain prediction error test fails at a specified instant, then a signal which is (in the limit) persistently exciting is applied. On the other hand, if the test is passed, then a particular certainty equivalence control law using the current estimate is applied. This strategy has common-sense appeal, despite the fact that the laws are somewhat complex in their present form. It is shown in the analysis of these laws that eventually the prediction error tests must always be passed, and hence it is shown that the system "locks on" to an acceptable control law. In summary, the adaptive control algorithms used in Zhang and Chen [1992] and Chen [1992] are as follows:

Step A) Introduce an appropriate criterion to judge whether or not the parameter estimate is satisfactory (for instance, a prediction error criterion).

Step B) Apply an excitation signal to the system, and estimate the unknown parameters via a least-squares (or related) algorithm until a "satisfactory" estimate is obtained according to the criterion; and after this,

Step C) construct a control law via the previously obtained "satisfactory" parameter estimates and use this law to control the system until some "unsatisfactory" property appears according to the criterion; and then

Step D) repeat this procedure through Steps B) and C).

If no "unsatisfactory" property appears at some stage in Step C), then the designed adaptive control law is used forever.

It is worth noticing that in some previous works (i) one or both of the derivatives dx_t and dy_t of the system state x and observation process y are required to be measurable in the parameter estimation procedure (see, e.g., Caines [1992]; Chen [1992];

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Chen and Guo [1990]; Chen and Moore [1987]; Duncan and Pasik-Duncan [1990], [1991]; Gevers, Goodwin, and Wertz [1991]; Goodwin, et al. [1991]; Moore [1988]; Christopeit [1986]); (ii) the criteria used in steps A through C have to be verified at all time instants, which is an uncountable procedure because of the nature of the continuous time model (see, e.g., Chen [1992] and Zhang and Chen [1991]); (iii): the unknown parameters are always estimated no matter whether they are needed or not (see, e.g., Chen [1992]; Chen and Zhang [1992]; and Zhang and Chen [1991]); (iv) some external stochastic excitation signals are invoked (see, e.g., Chen and Zhang [1992] and Zhang and Chen [1992]).

In this paper, we formulate an adaptive control algorithm which (i) avoids use of dx_t or dy_t and the introduction of external stochastic signals in the procedures of parameter estimation and adaptive control, (ii) simplifies the criteria in steps A through C so that they are required to be verified at discrete time instants only, (iii) stop the parameter estimation procedure when it is not needed in order to make the adaptive control law more practical, and finally, (iv) does not use an external stochastic excitation signal. (In effect the Brownian motion w driving the system is exploited for this purpose.)

It may be conjectured that such an alternation of identification and control regimes will work in certain time-varying cases.

2. Full observation systems. In this section we consider the LQ adaptive control problem for the following system model:

$$dx_t = Ax_t dt + Bu_t dt + C dw_t, \quad t \geq 0, \tag{2.1}$$

where $x_t \in \mathbb{R}^n$ and $u_t \in \mathbb{R}$ are the state and input of the system and $\{w_t, \mathcal{F}_t\}$ is a standard Wiener process in $\mathbb{R}^m$.

Using controls which at any instant t are based only on information available up to time t, we wish to stabilize the system (2.1). In this paper, this is achieved by the use of controls which are certain time-interleaved versions of an excitation signal and a signal designed via the certainty equivalence principle for the following quadratic loss function:

$$\min_{u \in \mathcal{U}} \limsup_{t \to \infty} J_t(u), \tag{2.2}$$

where

$$\mathcal{U} = \left\{ \{u_t : \; u_t \in \mathcal{F}_t \triangleq \sigma\{x_\tau, \; u_s, \; 0 \leq \tau \leq t, \; 0 \leq s < t\}, \; t \geq 0 \right\}, \tag{2.3}$$

$$J_t(u) = \frac{1}{t} \int_0^t (x_s^\tau Q_1 x_s + Q_2 u_s^2) ds, \quad Q_1 \geq 0, \quad Q_2 > 0. \tag{2.4}$$

This problem has been investigated in previous work; see, for instance, Zhang and Chen [1991], Chen [1992], where these authors presented the first rigorous stability analysis for such adaptive stabilization of a system, which might be neither open-loop stable nor minimum phase and might be subject to disturbances with an unknown bound.

Specification of the adaptive control law. First, we define a causal system which shall generate a disturbance input u', which shall be employed over an at most countable set of intervals; second, we define a linear state feedback control law, which shall be used over the intervals which interleave those during which the disturbance is used; then, third, we give a rule for determining the switching times which depend solely upon the history of the system inputs and outputs.

Following the procedure described from step A to step D, we now find an adaptive stabilization control for model (2.1).

Assume that the control input u is defined on the interval $[0, t)$; then the input u' and a countable sequence of stopping times with no finite accumulation point are defined as follows:

Let $T > 1$ and α be positive constants chosen arbitrarily. Define for $i = 1, 2, \ldots, n+1$

$$\beta_i = (-1)^{i+1}\alpha^i \frac{(n+1)!}{i! \times (n+1-i)!} \quad \text{with} \quad 0! \triangleq 1, \quad i! \triangleq 1 \times 2 \times \cdots \times i \tag{2.5}$$

and for some integer $k \geq 0$

$$u'_t = L_{T^k} + \beta_1 S_k u'_t + \cdots + \beta_{n+1} S_k^{n+1} u'_t, \quad t \in [T^k, \; T^{k+1}), \tag{2.6}$$

where $S_k u_t = \int_{T^k}^t u_s ds$ and $L_t = 1 + \int_0^t (\|\psi_s\|^2 + \|z_s\|^2 + u_s^2) ds$ with $\psi_t = [Sx_t^\tau, \; Su_t]^\tau$ and $z_t = [S\overline{x}_t^\tau, \; S\overline{u}_t]^\tau$. Here, S is the integral operator $Sx_t = \int_0^t x_s ds$; x_t and u_t are, respectively, the system state and system input, which is recursively given by (2.5)–(2.12); $\overline{x}_t$ and $\overline{u}_t$ are the solutions of the following equations, respectively:

$$(S+1)\overline{x}_t = x_t, \quad (S+1)\overline{u}_t = u_t,$$

i.e.,

$$\overline{x}_t = x_t - \int_0^t e^{-(t-\lambda)} x_\lambda d\lambda, \quad \overline{u}_t = u_t - \int_0^t e^{-(t-\lambda)} u_\lambda d\lambda.$$

It will be seen below that the function u which appears in the definitions above is equal to u' during the time intervals when the excitation input to the system is in use and is given as a linear function of the state x during the periods when u' is not being used as a system input.

Set $\theta = [A, \; B]^\tau$. Choosing an arbitrary θ_0, the unknown parameter θ is estimated via the least-squares method, which is modified so as to be active only over a sequence of intervals $[T^{\tau_{i-1}}, T^{\sigma_i})$. Specifically, the estimate $\theta_t = [A_t, \; B_t]^\tau$ is given by

$$\dot{\theta}_t = P_t \psi'_t(\overline{x}_t - \theta_t^\tau \psi'_t) \quad \text{with} \quad P_t = \left(I + \int_0^t \psi'_s \psi'^\tau_s ds\right)^{-1} \tag{2.7}$$

and

$$\psi'_t = \begin{cases} z_t & \text{if } t \in [0, T^{\tau_0}) \text{ or } t \in [T^{\tau_{i-1}}, T^{\sigma_i}) \\ 0 & \text{if } t \in [T^{\sigma_i}, T^{\tau_i}) \end{cases} \quad \forall i \geq 1, \tag{2.8}$$

where $\{\tau_i\}$ and $\{\sigma_i\}$ are two stopping time sequences defined as follows: $0 = \tau_0 < \sigma_1 < \tau_1 < \sigma_2 < \tau_2 < \cdots$,

$$\begin{aligned} \sigma_i = \inf\{k > \tau_{i-1} : \; & k \in \mathcal{N}, \quad (A_{T^k}, B_{T^k}, D) \text{ is controllable and observable,} \\ & \text{where here, and hereafter, } \mathcal{N} \text{ denotes the set of all positive} \\ & \text{integers, and } D \text{ is any square matrix such that } D^\tau D = Q_1\}, \end{aligned} \tag{2.9}$$

$$\tau_i = \inf\left\{k > \sigma_i : k \in \mathcal{N}, \quad \int_0^{T^k} \|x_s\|^2 ds > T^{\sigma_i} \int_0^{T^{\sigma_i}} (\|x_s\|^2 + \|u''_s\|^2) ds \right.$$
$$\left. + T^{\sigma_i} T^k + T^{\sigma_i} \right\} \tag{2.10}$$

with the excitation input u'' given by

$$u''_t = \begin{cases} u'_t & \text{if } t \in [0, T^{\tau_0}) \text{ or } t \in [T^{\tau_{i-1}}, T^{\sigma_i}) \\ 0 & \text{if } t \in [T^{\sigma_i}, T^{\tau_i}) \end{cases} \quad \forall i \geq 1. \tag{2.11}$$

Here Q_1 is given in (2.4) and u'_t is defined by (2.6).

The adaptive control u is generated by interleaving the excitation input u' and a linear feedback input as follows:

$$u_t = \begin{cases} u'_t & \text{if } t \in [0, T^{\tau_0}) \text{ or } t \in [T^{\tau_{i-1}}, T^{\sigma_i}) \text{ for some } i \geq 1, \\ -Q_2^{-1} B^{\tau}_{T^{\sigma_i}} R_{T^{\sigma_i}} x_t & \text{if } t \in [T^{\sigma_i}, T^{\tau_i}) \text{ for some } i \geq 1, \end{cases} \tag{2.12}$$

where Q_2 is the positive constant in (2.4), $B_{T^{\sigma_i}}$ is the estimate for B at time instant T^{σ_i} given by (2.7) and (2.8), and $R_{T^{\sigma_i}}$ is a solution of the following algebraic Riccati equation:

$$A^{\tau}_{T^{\sigma_i}} R_{T^{\sigma_i}} + R_{T^{\sigma_i}} A_{T^{\sigma_i}} - R_{T^{\sigma_i}} B_{T^{\sigma_i}} Q_2^{-1} B^{\tau}_{T^{\sigma_i}} R_{T^{\sigma_i}} + D^{\tau} D = 0.$$

Here $A_{T^{\sigma_i}}$ is the estimate for A at time instant T^{σ_i} given by (2.7) and (2.8).

Remark 2.1. From the definition (2.12) of u_t, it is easy to see that τ_i and σ_i are Markov times, i.e., $\sigma\{T^{\tau_i} \leq t\} \in \mathcal{F}_t$ and $\sigma\{T^{\sigma_i} \leq t\} \in \mathcal{F}_t$. Thus, $u \in \mathcal{U}$ and $\psi'_t \in \mathcal{F}_t$.

Remark 2.2. The excitation signal in (2.12) is generated by (2.6), in which a designer needs only determine the deterministic coefficients β_i. No additional stochastic signal is introduced except L_{T^k}, so we call this a deterministic-like excitation signal.

Remark 2.3. In (2.9), to get σ_i, the only thing one should do is to check the controllability and observability of (A_{T^k}, B_{T^k}, D) for every integer k at time instant T^k. While in (2.10), one need only check whether or not

$$\int_0^{T^k} \|x_s\|^2 ds > T^{\sigma_i} \int_0^{T^{\sigma_i}} (\|x_s\|^2 + \|u''_s\|^2) ds + T^{\sigma_i} T^k + T^{\sigma_i}$$

for every integer k at time instant T^k and such a set of time instants is evidently countable.

Remark 2.4. By (2.7) and (2.8) it is easy to see that $\theta_t = \theta_{T^{\sigma_i}}$ for all $t \in [T^{\sigma_i}, T^{\tau_i})$. In other words, unlike in Zhang and Chen [1991] or in Chen [1992], the LS parameter estimation is not carried out in the time interval $[T^{\sigma_i}, T^{\tau_i})$. Thus, if adaptive control (2.12) results in an integer i such that $\sigma_i < \infty$ and $\tau_i = \infty$, then the unknown parameter estimates will be locked on an acceptable value $\theta_{T^{\sigma_i}}$ forever.

LEMMA 2.1. *Let $\lambda^{(t)}_{\min}$ denote the smallest eigenvalue of matrix P_t^{-1}. Then the parameter estimate θ_t given by* (2.7)–(2.12) *has the following property:*

$$\|\theta_t - \theta\|^2 \leq \frac{c(t+1)}{\lambda^{(t)}_{\min}} \quad \forall t \geq 0,$$

where here and hereafter $c \geq 0$ is a possibly random quantity which is independent of t.

Proof. Let $\widetilde{\theta}_t = \theta_t - \theta$. Then from (2.1) it follows that

$$x_t = x_0 + \theta^\tau \psi_t + Cw_t.$$

Substituting this into the first equation of (2.7) and noting (2.8), $(S+1)\overline{x}_t = x_t$, and $(S+1)z_t = \psi_t$ we get

$$\dot{\widetilde{\theta}}_t = -P_t\psi_t'\psi_t'^\tau\widetilde{\theta}_t + P_t\psi_t'[(S+1)^{-1}(Cw_t) + \varepsilon_t],$$

where here and hereafter ε_t denotes a time function which exponentially converges to zero.

From this and the second equation of (2.7) we obtain

$$\frac{d(\widetilde{\theta}_t^\tau P_t^{-1}\widetilde{\theta}_t)}{dt} = -(\widetilde{\theta}_t^\tau\psi_t')^2 + 2\widetilde{\theta}_t^\tau\psi_t'[(S+1)^{-1}(Cw_t) + \varepsilon_t],$$

which implies that

$$\begin{aligned} 0 &\le \widetilde{\theta}_t^\tau P_t^{-1}\widetilde{\theta}_t \\ &= \widetilde{\theta}_0^\tau P_0^{-1}\widetilde{\theta}_0 - \int_0^t (\widetilde{\theta}_s^\tau\psi_s')^2 ds + 2\int_0^t \widetilde{\theta}_s^\tau\psi_s'[(S+1)^{-1}(Cw_s) + \varepsilon_s]ds \\ &\le \widetilde{\theta}_0^\tau P_0^{-1}\widetilde{\theta}_0 + \int_0^t [(S+1)^{-1}(Cw_s) + \varepsilon_s]^2 ds = O(t), \end{aligned}$$

where for the last inequality we have invoked $\int_0^t[(S+1)^{-1}(Cw_s)]^2ds = O(t+1)$ a.s. (e.g., Chen and Guo [1990]).

Therefore, Lemma 2.1 is true. □

LEMMA 2.2. *In system* (2.1), *if* (A, B) *is controllable and* $u_t = u_t^l$ *for some* k *and all* $t \in (T^k,\ T^{k+1}]$, *then there exist* $c > 0$, $a > 1$, *and* $k_0 > 0$ *such that*

$$\lambda_{\min}^{(T^{k+1})} \ge ca^{T^{k+1}} L_{T^k} \quad \forall k \ge k_0. \tag{2.13}$$

Proof. See Appendix A.

THEOREM 2.1. *If* Span$(B) \subset$ Span(C) *and* (A, B, D) *is controllable and observable with* $D^\tau D = Q_1$, *then under the adaptive control* (2.12) *it is the case that*

(a) *there is an integer* i *such that* $\sigma_i < \infty$, $\tau_i = \infty$, *and* $\theta_t = \theta_{T^{\sigma_i}}$ $\forall t \ge T^{\sigma_i}$;

(b) $\lim_{t\to\infty} \frac{1}{t}\int_0^t x_s x_s^\tau ds$ *and* $\lim_{t\to\infty}\frac{1}{t}\int_0^t u_s^2 ds$ *exist and are finite a.s.*

Here and hereafter by Span(X) *we mean the linear space spanned by the column vectors of* X.

Proof. Let R be the solution of the following algebraic Riccati equation:

$$A^\tau R + RA - RBQ_2^{-1}B^\tau R + D^\tau D = 0.$$

Then it is well known that $\Phi \triangleq A - BQ_2^{-1}B^\tau R$ is stable, and hence there is a positive matrix P such that

$$P\Phi + \Phi^\tau P = -I.$$

From this, it is easy to see that there exists a small enough positive constant ε such that

$$P\Phi_t + \Phi_t^\tau P \le -\frac{1}{2}I \quad \forall \Phi_t \in \{\Phi_t :\ \|\Phi_t - \Phi\| < \varepsilon\}. \tag{2.14}$$

We now define

$$\Phi_t = \begin{cases} A & \text{if } t \in [0, T^{\tau_0}) \text{ or } t \in [T^{\tau_{i-1}}, T^{\sigma_i}), \\ A - BQ_2^{-1}B_{T^{\sigma_i}}^{\tau} R_{T^{\sigma_i}} & \text{if } t \in [T^{\sigma_i}, T^{\tau_i}). \end{cases}$$

Then the adaptive control system defined above is expressed as

$$dx_t = \Phi_t x_t dt + Bu_t'' dt + C dw_t.$$

From the definitions (2.9) and (2.10) of sequences $\{\sigma_i\}$ and $\{\tau_i\}$ we see that only three cases can possibly hold. The first case is that there exists an integer i such that $\tau_{i-1} < \infty$ and $\sigma_i = \infty$; the second is that $\sigma_i < \tau_i < \infty$ for every integer $i \geq 1$, and the third is there exists an integer i such that $\sigma_i < \infty$ and $\tau_i = \infty$. We shall now show the first two cases are impossible.

Case 1. It is impossible that an integer i such that $\tau_{i-1} < \infty$ and $\sigma_i = \infty$ exists.

From Lemmas 2.1 and 2.2, it is easy to see that if there were an integer i such that $\tau_{i-1} < \infty$ and $\sigma_i = \infty$, then there would be

$$A_{T^k} \xrightarrow[k \to \infty]{} A \text{ and } B_{T^k} \xrightarrow[k \to \infty]{} B.$$

Thus, by the assumption that (A, B, D) is controllable and observable, we see that there would exist a k such that $T^k \geq T^{\tau_{i-1}}$ and (A_{T^k}, B_{T^k}, D) is controllable and observable. This contradicts $\sigma_i = \infty$.

Case 2. It is impossible that $\sigma_i < \tau_i < \infty$ for every integer $i \geq 1$.

If for every integer $i \geq 1$, $\sigma_i < \tau_i < \infty$, then by Lemmas 2.1 and 2.2 it is easy to see that $\Phi_{T^{\sigma_i}} \xrightarrow[i \to \infty]{} \Phi$. Therefore, there exists i_0 such that for all $i \geq i_0$,

$$\|\Phi_{T^{\sigma_i}} - \Phi\| < \varepsilon,$$

which together with (2.14) implies that

$$\|\Phi_t\| \leq c \quad \text{and} \quad P\Phi_{T^{\sigma_i}} + \Phi_{T^{\sigma_i}}^{\tau} P \leq -\frac{1}{2} I. \tag{2.15}$$

Using Ito's formula (cf. Schwartz [1984]) we find that for $k \in [\sigma_i, \tau_i) \cap \mathcal{N}$

$$\begin{aligned} x_{T^k}^{\tau} P x_{T^k} \leq x_0^{\tau} P x_0 &+ \int_0^{T^{\sigma_i}} x_s^{\tau}(P\Phi_s + \Phi_s^{\tau} P) x_s ds - \frac{1}{2} \int_{T^{\sigma_i}}^{T^k} \|x_s\|^2 ds \\ &+ 2\int_0^{T^{\sigma_i}} x_s^{\tau} P B u_s'' ds + 2\int_0^{T^k} x_s^{\tau} P C dw_t + \operatorname{tr}(C^{\tau} P C) T^k, \end{aligned} \tag{2.16}$$

where here and hereafter $\operatorname{tr}(X)$ denotes the trace of X.

Note that by Lemma 4 of Christopeit [1986], there exist random numbers c', c'', independent of t, such that for all k sufficiently large, say, for all $k \geq \sigma_m$,

$$\int_0^{T^k} x_s^{\tau} P C dw_s \leq c' \left(\int_0^{T^k} \|x_s^{\tau}\|^2 ds \right)^{\eta + \frac{1}{2}} + c' \leq c'' + \frac{1}{8} \int_0^{T^k} \|x_s^{\tau}\|^2 ds, \quad \eta \in \left(0, \frac{1}{2}\right).$$

Then from (2.16), for some random number c''' independent of time, we have

$$\begin{aligned} x_{T^k}^{\tau} P x_{T^k} \leq c''' &+ c''' \int_0^{T^{\sigma_m}} \|x_s\|^2 ds - \frac{3}{8} \int_{T^{\sigma_m}}^{T^k} \|x_s\|^2 ds \\ &+ c \int_0^{T^{\sigma_m}} \|u_s''\|^2 ds + \operatorname{tr}(C^{\tau} P C) T^k, \end{aligned}$$

and hence, for some random number c independent of time,

$$\int_0^{T^k} \|x_s\|^2 ds \leq c \int_0^{T^{\sigma_m}} (\|x_s\|^2 + \|u_s''\|^2) ds + cT^k + c \quad \forall k \geq \sigma_m. \tag{2.17}$$

Now there exists i sufficiently large that $T^{\sigma_i} \geq c$, and so by (2.17) this gives

$$\int_0^{T^k} \|x_s\|^2 ds \leq T^{\sigma_i} \int_0^{T^{\sigma_i}} (\|x_s\|^2 + \|u_s''\|^2) ds + T^{\sigma_i} T^k + T^{\sigma_i} \quad \forall k \geq \sigma_i,$$

which contradicts $\tau_i < \infty$.

So, there must exist an integer i such that $\sigma_i < \infty$ and $\tau_i = \infty$.

Thus by (2.7) and (2.8) we get Assertion (a) of Theorem 2.1. Furthermore, (2.10) together with $\tau_i = \infty$ implies that

$$\limsup_{k\to\infty} \frac{1}{T^k} \int_0^{T^k} \|x_s\|^2 ds < \infty \quad \text{a.s.} \tag{2.18}$$

Notice that $\text{Span}(B) \subset \text{Span}(C)$ and (A, B) is controllable implies that $(\Phi_{T^{\sigma_i}}, C)$ is controllable. Then by (2.18) and Lemma B.1 in Appendix B we see that $\Phi_{T^{\sigma_i}}$ is stable. Therefore, from Lemma 3 of Chen and Guo [1990] it follows that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t x_s x_s^\tau ds \quad \text{exists and is finite a.s.,}$$

which together with (2.12) implies that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t u_s^2 ds = Q_2^{-2} B_{T^{\sigma_i}}^\tau R_{T^{\sigma_i}} \left(\lim_{t\to\infty} \frac{1}{t} \int_0^t x_s x_s^\tau ds \right) (B_{T^{\sigma_i}}^\tau R_{T^{\sigma_i}})^\tau \text{ exists and is finite a.s.}$$

This proves Assertion (b) of Theorem 2.1. □

3. Partially observed system.

3.1. Problem statement. In this section we consider the single-input single-output continuous-time system described by

$$A(S) y_t = y_0 + SB(S) u_t + C(S) w_t + S\eta_t \quad \forall t \geq 0, \tag{3.1}$$

where $A(S)$, $B(S)$, and $C(S)$ are polynomials in S with unknown coefficients:

$$A(S) = 1 + \sum_{i=1}^p a_i S^i, \quad B(S) = \sum_{i=1}^q b_i S^{i-1}, \quad C(S) = \sum_{i=0}^l c_i S^i; \tag{3.2}$$

$\{w_t, \mathcal{F}_t\}$ is a standard Wiener process with respect to a nondecreasing σ-algebras $\{\mathcal{F}_t\}$ defined on a probability space; y_t and u_t are the system output and input, respectively, and measurable with respect to $\mathcal{F}_t$; and η_t is unknown disturbance or unmodeled dynamics which is measurable with respect to $\mathcal{F}_t$.

As Zhang and Chen (1991) show, model (3.1) subject to (3.2) is very general and includes some widely used models. For instance, in the case where $l = p$ and $c_p = ga_p$, the input-output properties of (3.1) are equivalent to those of the following well-known state-space representation (e.g., Gevers, Goodwin, and Wertz [1991]; Goodwin, et al. [1991]; Caines [1992]):

$$\begin{aligned} dx_t &= Ax_t dt + Bu_t dt + C dw_t + D\eta_t dt, \\ dy_t &= D^\tau x_t dt + g dw_t, \end{aligned}$$

with

$$A = \begin{bmatrix} -a_1 & 1 & & \\ -a_2 & & \ddots & \\ \vdots & & & 1 \\ -a_m & & & 0 \end{bmatrix}, \; B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}, \; C = \begin{bmatrix} c_0 - ga_0 \\ \vdots \\ c_{m-1} - ga_{m-1} \end{bmatrix}, \; D = \left.\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} m,$$

where $m = \max\{p, q\}$ and here and hereafter we set $a_0 = 1$, $a_i = 0$ for $i > p$, $b_j = 0$ for $j > q$, and $c_k = 0$ for $k > l$.

Let us denote the collection of unknown coefficients of $A(S)$ and $B(S)$ by θ:

$$\theta = [-a_1, \ldots, -a_p,\; b_1, \ldots, b_q]^\tau. \tag{3.3}$$

Let

$$F(S) = 1 + f_1 S + \cdots + f_{l+1} S^{l+1} \quad \text{with } f_{l+1} \neq 0$$

be an arbitrarily given stable polynomial of S; i.e., every S that satisfies $F(S) = 0$ has negative real part.

Denote by y_t^f and u_t^f the filtered value, respectively:

$$F(S) y_t^f = y_t, \quad F(S) u_t^f = u_t \tag{3.4}$$

and

$$\varphi_t^f = [S y_t^f, \ldots, S^p y_t^f, \quad S u_t^f, \ldots, S^q u_t^f]^\tau. \tag{3.5}$$

Define

$$\varphi_t' = \begin{cases} \varphi_t^f & \text{if } t \in [0, T^{\tau_0}) \text{ or } t \in [T^{\tau_{i-1}}, T^{\sigma_i}) \text{ for some } i \geq 1, \\ 0 & \text{if } t \in [T^{\sigma_i}, T^{\tau_i}) \text{ for some } i \geq 1, \end{cases} \tag{3.6}$$

where $\{\tau_i\}$ and $\{\sigma_i\}$ are two stopping time sequences such that $\varphi_t^f \in \mathcal{F}_t$.

Then the unknown parameter θ is estimated as follows:

$$\dot{\theta}_t = P_t \varphi_t' (y_t^f - \theta_t^\tau \varphi_t') \quad \text{with} \quad P_t = \left(I + \int_0^t \varphi_s' \varphi_s'^\tau ds \right)^{-1}, \tag{3.7}$$

where θ_0 is a constant chosen arbitrarily.

The purpose of this paper is to design a θ_t-based adaptive control so that the closed-loop system is stabilized in the sense that

$$\sup_{t \geq 0} \frac{1}{t+1} \int_0^t (y_s^2 + u_s^2) ds < \infty \quad \text{a.s.} \tag{3.8}$$

under the following assumptions:

A.1. $A(S)$ and $SB(S)$ are coprime, $b_q \neq 0$, $l \leq \min\{p, q-1\}$, and p and q are known.

A.2. $\sup_{t\geq 0} \frac{1}{t+1} \int_0^t \eta_s^2 ds < \infty$.

From Zhang and Chen [1991] it follows that Assumptions A.1 and A.2 are as weak as the following necessary and sufficient ones even when θ is known:

A.1$'$. The greatest common factor of $A(S)$ and $SB(S)$ is 1 or a stable polynomial, $b_q \neq 0$, the order of the greatest common factor, $l \leq \min\{p, q-1\}$, and p and q are known.

A.2$'$. $\sup_{t\geq 0} \frac{1}{t+1} \int_0^t \eta_s^2 ds < \infty$ a.s.

However, for simplicity of notation, in this paper we use Assumptions A.1 and A.2.

Remark 3.1. We now look at how to calculate the filtered values y_t^f and u_t^f of y_t and u_t, respectively, with respect to filter $F(S)$ in (3.4).

Let

$$D_F = \begin{bmatrix} -f_1 & -f_2 & \cdots & -f_{l+1} \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{bmatrix}, \quad Y_t = \begin{bmatrix} y_t^f \\ Sy_t^f \\ \vdots \\ S^l y_t^f \end{bmatrix}, \quad H_l = \left.\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} l+1.$$

Then from (3.4) we see that

$$Y_t = D_F S Y_t + H_l y_t,$$

which is equivalent to

$$Y_t = D_F \int_0^t e^{D_F(t-\lambda)} H_l y_\lambda d\lambda + H_l y_t.$$

Thus we have

$$y_t^f = H_l^\tau Y_t = y_t + H_l^\tau D_F \int_0^t e^{D_F(t-\lambda)} H_l y_\lambda d\lambda.$$

Similarly, we can get

$$u_t^f = u_t + H_l^\tau D_F \int_0^t e^{D_F(t-\lambda)} H_l u_\lambda d\lambda.$$

3.2. Adaptive control. We first look at what the stabilization control is in the case where θ is known. To see this, we introduce the following lemma (e.g., Chen and Guo [1990]).

LEMMA 3.1. *Let $k \geq 0$ be an integer and $E(S) = 1 + e_1 S + \cdots + e_k S^k$ be a stable polynomial with $e_k \neq 0$. Then there is a (nonrandom) constant $C_e \geq 1$ (depending on $E(S)$ only) such that*

$$\sum_{i=0}^k \int_0^t \left(\frac{S^i}{E(S)} z_\lambda \right)^2 d\lambda \leq C_e \int_0^t z_\lambda^2 d\lambda$$

for any square-integrable process $\{z_t\}$.

If $A(S)$, $SB(S)$ are coprime and $b_q \neq 0$, then for any polynomial

$$E(S) = 1 + e_1 S + \cdots + e_{p+q} S^{p+q} \quad \text{with} \quad e_{p+q} \neq 0, \tag{3.9}$$

there exists a unique polynomial pair $(G(S),\ H(S))$ such that

(3.10) $A(S)G(S) - SB(S)H(S) = E(S)$ with $\partial(G(S)) \le q-1$ and $\partial(H(S)) = p$,

where here and hereafter $\partial(X(S))$ denotes the degree of the polynomial $X(S)$ in S.

From (3.10) and (3.1) it is clear that

$$\begin{aligned} E(S)y_t &= A(S)G(S)y_t - SB(S)H(S)y_t \\ &= G(S)[A(S)y_t - SB(S)u_t] + SB(S)[G(S)u_t - H(S)y_t] \\ &= G(S)[y_0 + C(S)w_t + S\eta_t] + SB(S)[G(S)u_t - H(S)y_t] \end{aligned}$$

and

$$\begin{aligned} E(S)u_t &= A(S)G(S)u_t - SB(S)H(S)u_t \\ &= H(S)[A(S)y_t - SB(S)u_t] + A(S)[G(S)u_t - H(S)y_t] \\ &= H(S)[y_0 + C(S)w_t + S\eta_t] + A(S)[G(S)u_t - H(S)y_t]. \end{aligned}$$

Therefore, in the case where θ is known, $l \le \min\{p,\ q-1\}$ and Assumptions A.1 and A.2 hold, for any given stable $E(S)$ subject to (3.9), if the control is defined as follows:

$$G(S)u_t - H(S)y_t = 0, \quad t \ge 0,$$

then the system is stabilized in the average sense (3.8).

Similar to §2, we now introduce a deterministic-like excitation signal u'_t. Let $T > 1$ and α be positive constants chosen arbitrarily. Define for $i = 1, 2, \ldots, p+q$,

$$\beta_i = (-1)^{i+1}\alpha^i \frac{(p+q)!}{i! \times (p+q-i)!} \quad \text{with} \quad 0! \triangleq 1,\ \ i! \triangleq 1 \times 2 \times \cdots \times i,$$

and for some integer $k \ge 0$,

$$u'_t = L_{T^k} + \beta_1 S_k u'_t + \cdots + \beta_{p+q} S_k^{p+q} u'_t, \quad t \in [T^k,\ T^{k+1}),$$

where $S_k u_t = \int_{T^k}^t u_s ds$ and $L_t = 1 + \int_0^t (\|\varphi_s^f\|^2 + \|\varphi_s\|^2 + u_s^2) ds$ with φ_t^f defined by (3.5) and φ_t defined by

$$\varphi_t = [Sy_t, \ldots, S^p y_t, \quad Su_t, \ldots, S^q u_t]^\tau. \tag{3.11}$$

For any $k \ge 1$, write θ_{T^k} in the component form

$$\theta_{T^k} = [-a_{1T^k}, \ldots, -a_{pT^k}, \quad b_{1T^k}, \ldots, b_{qT^k}]^\tau$$

and set

$$A_k(S) = 1 + \sum_{i=1}^p a_{iT^k} S^i, \quad B_k(S) = \sum_{i=1}^q b_{iT^k} S^{i-1}. \tag{3.12}$$

Let $E(S)$ be a stable polynomial subject to (3.9). Then by Lemma 3.1 there is a constant C_e (depending on $E(S)$ only) such that

$$\sum_{i=0}^{p+q} \int^t \left(\frac{S^i}{E(S)} z_\lambda\right)^2 d\lambda \le C_e \int_0^t z_\lambda^2 d\lambda \tag{3.13}$$

for any square-integrable process $\{z_\lambda\}$.

Let $R(S)$ be a stable polynomial of S with $\partial(R(S)) = \min\{p+1, q\}$ and let ξ_t^u, ξ_t^y denote the filtered values of u_t, y_t with respect to $R(S)$, respectively:

$$R(S)\xi_t^u = u_t, \quad R(S)\xi_t^y = y_t \quad \forall t \geq 0. \tag{3.14}$$

Actually, the filtered values ξ_t^u and ξ_t^y can be calculated as in Remark 3.1.

Set $\zeta_t = [S\xi_t^y, \ldots, S^p\xi_t^y,\ S\xi_t^u, \ldots, S^q\xi_t^u]^\tau$. Then by (3.11) and (3.14) we get

$$R(S)\zeta_t = \varphi_t. \tag{3.15}$$

In the following discussion, for a given polynomial $Z(S) = z_0 + z_1 S + \cdots + z_r S^r$, its norm is defined as

$$\|Z(S)\| = \left(\sum_{i=0}^r z_i^2\right)^{1/2}.$$

Define switching times $1 = \tau_0 < \sigma_1 < \tau_1 < \sigma_2 < \tau_2 \cdots$ as follows:

$$\begin{aligned} \sigma_i = \inf\Bigg\{ k > \tau_{i-1} :\ & A_k(S)G_k(S) - SB_k(S)H_k(S) = E(S) \text{ is solvable} \\ & \text{with respect to } G_k(S) \text{ and } H_k(S) \text{ subject to} \\ & \partial(G_k(S)) \leq q-1 \quad \text{and} \quad \partial(H_k(S)) = p; \quad \text{and} \\ & \|G_k(S)\|^2 + \|H_k(S)\|^2 \leq \frac{k}{2C_e(p+q+1)}; \\ & \int_0^{T^k} [\xi_s^y - \theta_{T^k}^\tau \zeta_s]^2\, ds \leq k^{-2} f(k, T^k) \Bigg\}, \end{aligned} \tag{3.16}$$

$$\begin{aligned} \tau_i = \min\Bigg\{ k > \sigma_i :\ & \text{there exists } t \in (T^{\sigma_i}, T^k] \text{ such that} \\ & \int_0^t [\xi_s^y - \theta_{T^{\sigma_i}}^\tau \zeta_s]^2\, ds > \sigma_i^{-2} f(\sigma_i, t) \Bigg\}, \end{aligned} \tag{3.17}$$

where C_e is given in (3.13), and

$$f(x,t) = (t+1) \sup_{0 \leq \lambda \leq t} \left\{ x^3 + \frac{1}{\lambda+1} \int_0^\lambda \left[\sum_{j=0}^{p+1} (S^j \xi_s^y)^2 + \sum_{j=0}^{q} (S^j \xi_s^u)^2 \right] ds \right\}. \tag{3.18}$$

Similar to (2.12) we define the adaptive control u_t as follows:

$$u_t = \begin{cases} u_t' & \text{if } t \in [0, T^{\tau_0}) \text{ or } t \in [T^{\tau_{i-1}}, T^{\sigma_i}) \text{ for some } i \geq 1, \\ H_{\sigma_i}(S)y_t - (G_{\sigma_i}(S) - 1)u_t & \text{if } t \in [T^{\sigma_i}, T^{\tau_i}) \text{ for some } i \geq 1, \end{cases} \tag{3.19}$$

where $H_{\sigma_i}(S)$ and $G_{\sigma_i}(S)$ are generated by (3.16), (3.12), and (3.4)–(3.7).

In this case, similar to Lemmas 2.1 and 2.2 we may obtain the following results.

LEMMA 3.2. *Let $\lambda_{\min}^{(t)}$ denote the smallest eigenvalue of matrix R_t^{-1}. Then the parameter estimate θ_t given by (3.4)–(3.7) has the following property:*

$$\|\theta_t - \theta\|^2 \leq \frac{c(t+1)}{\lambda_{\min}^{(t)}} \quad \forall t \geq 0$$

for some time-independent random variable $c \geq 0$.

Proof. Let $\widetilde{\theta}_t = \theta_t - \theta$. Then from (3.1)–(3.3) and (3.5) it follows that

$$y_t^f = \theta^\tau \varphi_t^f + F^{-1}(S)[C(S)w_t + S\eta_t] + \varepsilon_t,$$

where we recall that ε_t denotes a function of time decaying exponentially to zero. Substituting this into the first equation of (3.7) and noting (3.6), we get

$$\dot{\widetilde{\theta}}_t = -P_t\varphi_t'\varphi_t'^\tau\widetilde{\theta}_t + P_t\varphi_t'\left[F^{-1}(S)(C(S)w_t + S\eta_t) + \varepsilon_t\right].$$

Then combining this with the second equation of (3.7) gives

$$\frac{d(\widetilde{\theta}_t^\tau P_t^{-1}\widetilde{\theta}_t)}{dt} = -(\widetilde{\theta}_t^\tau\varphi_t')^2 + 2\widetilde{\theta}_t^\tau\varphi_t'\left[F^{-1}(S)(C(S)w_t + S\eta_t) + \varepsilon_t\right],$$

which together with $\int_0^t[F^{-1}(S)(C(S)w_s]^2 ds = O(t)$ (cf. Lemma 3 of Chen and Guo [1990]) implies that

$$\begin{aligned} 0 &\leq \widetilde{\theta}_t^\tau P_t^{-1}\widetilde{\theta}_t \\ &\leq \widetilde{\theta}_0^\tau P_0^{-1}\widetilde{\theta}_0 - \int_0^t (\widetilde{\theta}_s^\tau\varphi_s')^2 ds + 2\int_0^t \widetilde{\theta}_s^\tau\varphi_s'\left[F^{-1}(S)(C(S)w_s + S\eta_s) + \varepsilon_t\right] ds \\ &\leq \widetilde{\theta}_0^\tau P_0^{-1}\widetilde{\theta}_0 + \int_0^t \left[F^{-1}(S)(C(S)w_s + S\eta_s) + \varepsilon_t\right]^2 ds = O(t+1), \end{aligned}$$

where for the final bound we have used Lemma 3.1 and Assumption A.2. □

LEMMA 3.3. *Under Assumptions* A.1 *and* A.2, *if* $u_t = u_t'$ *for some* k *and any* $t \in (T^k,\ T^{k+1}]$, *then there exist* $c > 0$, $a > 1$, *and* $k_0 > 0$ *such that*

$$\lambda_{\min}^{(T^{k+1})} \geq c a^{T^{k+1}} L_{T^k} \quad \forall k \geq k_0.$$

Proof. The proof resembles that of Lemma 2.2 and is given in Appendix C.

THEOREM 3.1. *Under Assumptions* A.1 *and* A.2 *and the adaptive control* (3.4)–(3.7), (3.16)–(3.19), *we get that*

(a) *there is an integer* i *such that* $\sigma_i < \infty$, $\tau_i = \infty$, *and* $\theta_t = \theta_{T^{\sigma_i}}$ $\forall t \geq T^{\sigma_i}$;

(b) $\sup_{t\geq 0} \frac{1}{t+1}\int_0^t (y_s^2 + u_s^2)ds < \infty$ *a.s.*

Proof. We first show that it is impossible that $\tau_i < \infty$ and $\sigma_{i+1} = \infty$ on a sample set $\mathcal{D}$ with positive probability for an integer-valued random variable $i \geq 0$.

In fact, if there were a sample set $\mathcal{D}$ of positive probability, i.e., for which $P(\mathcal{D}) > 0$, which was such that for every sample $\omega \in \mathcal{D}$, there were an $i(\omega) \geq 0$ (for simplicity, we drop ω below) such that $\tau_i < \infty$ and $\sigma_{i+1} = \infty$, then $u_t = u_t'$ for all $t \geq \tau_i$. Thus, by Lemmas 3.2 and 3.3 we would have that for some constant $a > 1$

$$\|\theta_{T^k} - \theta\|^2 = O\left(\frac{T^k}{a^{T^k} L_{T^k}}\right) = O\left(\frac{1}{k^3}\right) \quad \text{a.s. on } \mathcal{D} \quad \forall k > \tau_i, \tag{3.20}$$

which together with Lemma D.1 in Appendix D implies that there exists an integer $k_1 \geq 0$ such that for any $k \geq k_1$, $A_k(S)G_k(S) - SB_k(S)H_k(S) = E(S)$ is solvable with respect to $G_k(S)$ and $H_k(S)$ subject to $\partial(G_k(S)) \leq q-1$ and $\partial(H_k(S)) = p$, and $\|G_k(S)\|^2 + \|H_k(S)\|^2 \leq k/(2C_e(p+q+1))$.

From Lemma 3 of Chen and Guo [1990] and the fact that $\partial(R(S)) \geq \partial(C(S))+1$ it follows that

$$\int_0^t \left(\frac{C(S)}{R(S)} w_s\right)^2 ds = O(t) \quad \text{a.s.},$$

while from (3.1), (3.3), (3.14), and (3.15) it follows that

$$\xi_s^y - \theta_t^\tau \zeta_s = (\theta - \theta_t)^\tau \zeta_s + \frac{C(S)}{R(S)} w_s + \frac{S}{R(S)} \eta_s + \varepsilon_t \quad \forall t,\ s \geq 0.$$

Therefore, by Assumption A.2, Lemma 3.1, and (3.18) we find that

$$\begin{aligned}
&\frac{1}{f(k, T^k)} \int_0^{T^k} [\xi_s^y - \theta_{T^k}^\tau \zeta_s]^2 \, ds \\
&\leq \frac{4}{f(k, T^k)} \left[\int_0^{T^k} ((\theta - \theta_{T^k})^\tau \zeta_s)^2 \, ds + \int_0^{T^k} \left(\frac{C(S)}{R(S)} w_s\right)^2 ds \right. \\
&\qquad \left. + \int_0^{T^k} \left(\frac{S}{R(S)} \eta_s\right)^2 ds + \int_0^{T^k} (\varepsilon_s)^2 \, ds \right] \\
&= O\left(\|\theta_{T^k} - \theta\|^2 + \frac{1}{k^3} \right) = O\left(\frac{1}{k^3}\right) \quad \text{a.s. on } \mathcal{D},
\end{aligned} \tag{3.21}$$

where (3.20) is invoked for the last equality.

From (3.21) we conclude that there exists an integer $k_2 \geq k_1$ such that for any $k \geq k_2$,

$$\frac{1}{f(k, T^k)} \int_0^{T^k} [\xi_s^y - \theta_{T^k}^\tau \zeta_s]^2 \, ds \leq k^{-2} \quad \text{a.s. on } \mathcal{D}. \tag{3.22}$$

Thus, $\sigma_{i+1} < \infty$ a.s. on $\mathcal{D}$. This contradicts $\sigma_{i+1} = \infty$ on $\mathcal{D}$ and $P(\mathcal{D}) > 0$.

We now prove that $\tau_i = \infty$ a.s. for some integer-valued random variable $i \geq 1$.

In fact, from Lemmas 3.2 and 3.3 it follows that for some $a > 1$,

$$\|\theta_{T^{\sigma_i}} - \theta\|^2 = O\left(\frac{T^{\sigma_i}}{a^{T^{\sigma_i}} L_{T^{\sigma_i}}}\right) = O\left(\frac{1}{\sigma_i^3}\right). \tag{3.23}$$

As in (3.21) we would have

$$\frac{1}{f(\sigma_i, t)} \int_0^t [\xi_s^y - \theta_{T^{\sigma_i}}^\tau \zeta_s]^2 \, ds = O\left(\|\theta_{T^{\sigma_i}} - \theta\|^2 + \frac{1}{\sigma_i^3} \right) = O\left(\frac{1}{\sigma_i^3}\right) \leq \sigma_i^{-2},$$

where the last inequality is valid for some large enough i and $t \geq T^{\sigma_i}$ because of (3.23). Hence there must be $\tau_i = \infty$ for some i; i.e., assertion (a) is true. We now prove assertion (b). From assertion (a) and (3.19) it follows that for some $i \geq 1$,

$$H_{\sigma_i}(S) y_t - G_{\sigma_i}(S) u_t = 0, \quad t \geq T^{\sigma_i}. \tag{3.24}$$

Henceforth, for simplicity of notation, we shall write $\theta_{\sigma_i}^\tau$ for $\theta_{T^k}^\tau$.

In view of (3.16) we get

$$\begin{aligned} E(S)S^k y_t &= S^k A_{\sigma_i}(S)G_{\sigma_i}(S)y_t - S^{k+1}B_{\sigma_i}(S)H_{\sigma_i}(S)y_t \\ &= S^k G_{\sigma_i}(S)[A_{\sigma_i}(S)y_t - SB_{\sigma_i}(S)u_t] \\ &\quad + S^{k+1}B_{\sigma_i}(S)[G_{\sigma_i}(S)u_t - H_{\sigma_i}(S)y_t], \quad k = 0,\ 1, \ldots, p+1; \end{aligned} \tag{3.25}$$

$$\begin{aligned} E(S)S^k u_t &= S^k H_{\sigma_i}(S)[A_{\sigma_i}(S)y_t - SB_{\sigma_i}(S)u_t] \\ &\quad + S^k A_{\sigma_i}(S)[G_{\sigma_i}(S)u_t - H_{\sigma_i}(S)y_t], \quad k = 0,\ 1, \ldots, q. \end{aligned} \tag{3.26}$$

Thus, noting that $A_{\sigma_i}(S)y_t - SB_{\sigma_i}(S)u_t = y_t - \theta^\tau_{\sigma_i} S\varphi_t$, by (3.14), (3.25), and inequality $(a+b)^2 \le 2a^2 + 2b^2$ we get

$$\begin{aligned} S^j \xi^y_t &= R(S)^{-1} S^j y_t \\ &= E(S)^{-1} S^j G_{\sigma_i}(S) R(S)^{-1}(y_t - \theta^\tau_{\sigma_i}\varphi_t) \\ &\quad + E(S)^{-1} R(S)^{-1} S^{j+1} B_{\sigma_i}(S)[G_{\sigma_i}(S)u_t - H_{\sigma_i}(S)y_t] \\ &= E(S)^{-1} S^j G_{\sigma_i}(S)(\xi^y_t - \theta^\tau_{\sigma_i}\zeta_t) \\ &\quad + E(S)^{-1} R(S)^{-1} S^{j+1} B_{\sigma_i}(S)[G_{\sigma_i}(S)u_t - H_{\sigma_i}(S)y_t] \\ &\quad j = 0,\ 1, \ldots, p+1, \end{aligned}$$

and, furthermore, we have

$$\begin{aligned} \sum_{j=0}^{p+1}(S^j\xi^y_s)^2 &\le 2\|G_{\sigma_i}(S)\|^2 \sum_{j=0}^{p+1}\sum_{k=0}^{q-1}[S^{j+k}E^{-1}(S)(\xi^y_s - \theta^\tau_{\sigma_i}\zeta_s)]^2 \\ &\quad + 2\sum_{j=0}^{p+1}\left(E^{-1}(S)R^{-1}(S)S^{j+1}B_{\sigma_i}(S)[G_{\sigma_i}(S)u_s - H_{\sigma_i}(S)y_s]\right)^2 \\ &\le 2\|G_{\sigma_i}(S)\|^2(p+q+1)\sum_{j=0}^{p+q}[S^jE^{-1}(S)(\xi^y_s - \theta^\tau_{\sigma_i}\zeta_s)]^2 \\ &\quad + 2\sum_{j=0}^{p+1}\left(E^{-1}(S)R^{-1}(S)S^{j+1}B_{\sigma_i}(S)[G_{\sigma_i}(S)u_s - H_{\sigma_i}(S)y_s]\right)^2, \end{aligned} \tag{3.27}$$

and similarly, by (3.26) we get

$$\begin{aligned} \sum_{j=0}^{q}(S^j\xi^u_s)^2 &\le 2\|H_{\sigma_i}(S)\|^2 \sum_{j=0}^{q}\sum_{k=0}^{p}[S^{j+k}E^{-1}(S)(\xi^y_s - \theta^\tau_{\sigma_i}\zeta_s)]^2 \\ &\quad + 2\sum_{j=0}^{q}\left(E^{-1}(S)R^{-1}(S)S^jA_{\sigma_i}(S)[G_{\sigma_i}(S)u_s - H_{\sigma_i}(S)y_s]\right)^2 \\ &\le 2\|H_{\sigma_i}(S)\|^2(p+q+1)\sum_{j=0}^{p+q}[S^jE^{-1}(S)(\xi^y_s - \theta^\tau_{\sigma_i}\zeta_s)]^2 \\ &\quad + 2\sum_{j=0}^{q}\left(E^{-1}(S)R^{-1}(S)S^jA_{\sigma_i}(S)[G_{\sigma_i}(S)u_s - H_{\sigma_i}(S)y_s]\right)^2. \end{aligned} \tag{3.28}$$

By Lemma 3.1 and (3.24) we see that

$$\limsup_{t\to\infty}\frac{1}{t+1}\sum_{j=0}^{p+1}\int_0^t\left(E^{-1}(S)R^{-1}(S)S^{j+1}B_{\sigma_i}(S)[G_{\sigma_i}(S)u_s - H_{\sigma_i}(S)y_s]\right)^2 ds < \infty \quad \text{a.s.}$$

and that

$$\limsup_{t\to\infty} \frac{1}{t+1} \sum_{j=0}^{q} \int_0^t \left(E^{-1}(S)R^{-1}(S)S^j A_{\sigma_i}(S)[G_{\sigma_i}(S)u_s - H_{\sigma_i}(S)y_s]\right)^2 ds < \infty \quad \text{a.s.}$$

Therefore, by (3.27), (3.28), and Lemma 3.1 we conclude that for some $\nu_1 < \infty$ which is independent of t

(3.29)

$$\begin{aligned}
&\frac{1}{t+1}\int_0^t \left[\sum_{k=0}^{p+1}(S^k\xi_s^y)^2 + \sum_{k=0}^{q}(S^k\xi_s^u)^2\right] ds \\
&\le 2(p+q+1)[\|G_{\sigma_i}(S)\|^2 + \|H_{\sigma_i}(S)\|^2]\frac{1}{t+1}\sum_{j=0}^{p+q}\int_0^t \left[\frac{S^j}{E(S)}(\xi_s^y - \theta_{\sigma_i}^\tau \zeta_s)\right]^2 ds + \nu_1 \\
&\le \frac{2(p+q+1)\sigma_i}{2C_e(p+q+1)} \cdot \frac{C_e}{t+1}\int_0^t (\xi_s^y - \theta_{\sigma_i}^\tau\zeta_s)^2 ds + \nu_1 \\
&\le \sigma_i \cdot \sigma_i^{-2}\frac{1}{t+1}f(\sigma_i, t) + \nu_1 \quad \text{a.s.,} \quad t \ge T^{\sigma_i},
\end{aligned}$$

where (3.17), $\sigma_i < \infty$, and $\tau_i = \infty$ a.s. have been used for the last inequality.

Set

$$\nu_2 = \nu_1 + \sup_{0\le\lambda\le T^{\sigma_i}}\left\{\frac{1}{\lambda+1}\int_0^\lambda \left[\sum_{k=0}^{p+1}(S^k\xi_s^y)^2 + \sum_{k=0}^{q}(S^k\xi_s^u)^2\right] ds\right\}.$$

Then from (3.29) and (3.18) it follows that

$$\begin{aligned}
&\sup_{0\le\lambda\le t}\left\{\frac{1}{\lambda+1}\int_0^\lambda \left[\sum_{k=0}^{p+1}(S^k\xi_s^y)^2 + \sum_{k=0}^{q}(S^k\xi_s^u)^2\right] ds\right\} \le \sigma_i^{-1}\frac{1}{t+1}f(\sigma_i,t) + \nu_2 \\
&\le \sigma_i^2 + \nu_2 + \sigma_i^{-1}\sup_{0\le\lambda\le t}\left\{\frac{1}{\lambda+1}\int_0^\lambda \left[\sum_{k=0}^{p+1}(S^k\xi_s^y)^2 + \sum_{k=0}^{q}(S^k\xi_s^u)^2\right] ds\right\} \quad \text{a.s.,}
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\limsup_{t\to\infty}\sup_{0\le\lambda\le t}\left\{\frac{1}{\lambda+1}\int_0^\lambda \left[\sum_{k=0}^{p+1}(S^k\xi_s^y)^2 + \sum_{k=0}^{q}(S^k\xi_s^u)^2\right] ds\right\} \\
&\le (1-\sigma_i^{-1})^{-1}\left[\nu_2 + \sigma_i^2\right] < \infty \quad \text{a.s.}
\end{aligned}$$

From this and (3.14), assertion (b) follows. □

Appendix A. Proof of Lemma 2.2. Let

(A.1)
$$N = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n+1} \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{bmatrix}, \quad H = [1, \underbrace{0,\ldots,0}_{n}]^\tau,$$

and, for any $t \in [T^k,\ T^{k+1})$,

$$U_t(k) = [u_t,\ S_k u_t, \ldots, S_k^n u_t]^\tau.$$

Then from the definition of u_t it follows that for any $t \in (T^k,\ T^{k+1})$,

$$\frac{dU_t(k)}{dt} = NU_t(k) \quad \text{with } U_{T^k}(k) = HL_{T^k}, \tag{A.2}$$

i.e.,

$$S_k U_t(k) = N^{-1}U_t(k) - N^{-1}HL_{T^k} \quad \forall t \in [T^k,\ T^{k+1}). \tag{A.3}$$

We first to show that there exist constants $c > 0$, $\gamma > 1$ and k_0 such that

$$\lambda_{\min}\left(\int_{T^k}^{T^{k+1}} (S_k U_s(k))(S_k U_s(k))^\tau ds\right) \geq c\gamma^{T^{k+1}} L_{T^k}^2 \quad \forall k \geq k_0, \tag{A.4}$$

where here and hereafter $\lambda_{\min}(X)$ denotes the minimal eigenvalue of matrix X.

From (2.5) and (A.1) we see that the characteristic polynomial $\det(xI - N) = (x - \alpha)^{n+1}$ of N coincides with the minimal polynomial of N. Thus there is a nonsingular $(n+1) \times (n+1)$ matrix P such that

$$\overline{\Lambda} \triangleq P^{-1}NP = \begin{bmatrix} \alpha & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \alpha \end{bmatrix}_{(n+1)\times(n+1)}. \tag{A.5}$$

Let $\overline{U}_t(k) = P^{-1}U_t(k)$ and $\overline{H} = P^{-1}H$. Then (A.2) is equivalent to

$$\frac{d\overline{U}_t(k)}{dt} = \overline{\Lambda} \cdot \overline{U}_t(k) \quad \text{with } \overline{U}_{T^k}(k) = \overline{H}L_{T^k} \quad \forall t \in (T^k,\ T^{k+1}). \tag{A.6}$$

Noting that for a given positive semidefinite matrix U,

$$\lambda_{\min}(PUP^\tau) \geq \lambda_{\min}(PP^\tau)\lambda_{\min}(U)$$

and $\lambda_{\max}(PP^\tau) \leq \|P\|^2$, by (A.3) and inequality $a^2 \geq \frac{1}{2}b^2 - (b-a)^2$ we have

$$\begin{aligned}
&\lambda_{\min}\left(\int_{T^k}^{T^{k+1}} (S_k U_s(k))(S_k U_s(k))^\tau ds\right) \\
&\geq \frac{1}{2}\lambda_{\min}(N^{-1}N^{-\tau})\lambda_{\min}\left(\int_{T^k}^{T^{k+1}} U_s(k)U_s^\tau(k) ds\right) - \|N^{-1}\|^2 T^{k+1} L_{T^k}^2 \\
&\leq \frac{1}{2}\lambda_{\min}(N^{-1}N^{-\tau})\lambda_{\min}\left(PP^\tau\right)\lambda_{\min}\left(\int_{T^k}^{T^{k+1}} \overline{U}_s(k)\overline{U}_s^\tau(k) ds\right) - \|N^{-1}\|^2 T^{k+1} L_{T^k}^2.
\end{aligned}$$

Thus, in order to show (A.4), it suffices to prove that there exist constants $c > 0$, $\gamma > 1$, and k_0 such that

$$\lambda_{\min}\left(\int_{T^k}^{T^{k+1}} \overline{U}_s(k)\overline{U}_s^\tau(k) ds\right) \geq c\gamma^{T^{k+1}} L_{T^k}^2 \quad \forall k \geq k_0. \tag{A.7}$$

From (A.6) it follows that

$$\overline{U}_t(k) = e^{\overline{\Lambda}t}\overline{H}L_{T^k} \quad \forall t \in [T^k, T^{k+1}),$$

which implies that

$$\begin{aligned}
\int_{T^k}^{T^{k+1}} \overline{U}_s(k)\overline{U}_s^\tau(k)ds &= L_{T^k}^2 \int_{T^k}^{T^{k+1}} e^{\overline{\Lambda}s}\overline{H}\cdot\overline{H}^\tau e^{\overline{\Lambda}^\tau s}ds \\
&\geq L_{T^k}^2 \int_{T^{k+1}-1}^{T^{k+1}} e^{\overline{\Lambda}s}\overline{H}\cdot\overline{H}^\tau e^{\overline{\Lambda}^\tau s}ds \\
&= L_{T^k}^2 e^{\overline{\Lambda}(T^{k+1}-1)}\left(\int_0^1 e^{\overline{\Lambda}s}\overline{H}\cdot\overline{H}^\tau e^{\overline{\Lambda}^\tau s}ds\right)e^{\overline{\Lambda}^\tau(T^{k+1}-1)} \\
&\geq L_{T^k}^2 \lambda_{\min}\left(\int_0^1 e^{\overline{\Lambda}s}\overline{H}\cdot\overline{H}^\tau e^{\overline{\Lambda}^\tau s}ds\right)e^{\overline{\Lambda}(T^{k+1}-1)}e^{\overline{\Lambda}^\tau(T^{k+1}-1)}. \qquad \text{(A.8)}
\end{aligned}$$

Note that (N, H) is controllable and hence $(\overline{\Lambda}, \overline{H})$ is controllable. Therefore,

$$\lambda_{\min}\left(\int_0^1 e^{\overline{\Lambda}s}\overline{H}\cdot\overline{H}^\tau e^{\overline{\Lambda}^\tau s}ds\right) > 0. \qquad \text{(A.9)}$$

Set

$$\Sigma_t = \begin{bmatrix} 1 & & & \\ (t-1) & \ddots & & \\ \vdots & \ddots & \ddots & \\ \frac{(t-1)^n}{n!} & \cdots & (t-1) & 1 \end{bmatrix}.$$

Then from (A.5) we see that

$$e^{\overline{\Lambda}(t-1)} = e^{\alpha(t-1)}\Sigma_t. \qquad \text{(A.10)}$$

It is evident that

$$\det(\Sigma_t\Sigma_t^\tau) = 1 \quad \text{and} \quad \lambda_{\max}(\Sigma_t\Sigma_t^\tau) \leq (n+1)\sum_{i=0}^n (t-1)^{2i},$$

where $\lambda_{\max}(X)$ denotes the maximum eigenvalues of X.

Thus, from the fact that $\det(X) = \prod_{i=1}^{n+1}\lambda_i(X)$ for any $(n+1)\times(n+1)$ matrix with eigenvalues $\lambda_i(X)$ $(i = 1, \ldots, n+1)$ it follows that

$$\begin{aligned}
\lambda_{\min}(\Sigma_t\Sigma_t^\tau) \geq [\lambda_{\max}(\Sigma_t\Sigma_t^\tau)]^{-n} &\geq \left[(n+1)\sum_{i=0}^n (t-1)^{2i}\right]^{-n} \\
&\geq (n+1)^{-2n}(t-1)^{-2n^2} \quad \forall t \geq 2.
\end{aligned}$$

From this and (A.10) we get that

$$\lambda_{\min}\left(e^{\overline{\Lambda}(t-1)}e^{\overline{\Lambda}^\tau(t-1)}\right) \geq e^{2\alpha(t-1)}\cdot(n+1)^{-2n}(t-1)^{-2n^2} \quad \forall t \geq 2,$$

which together with $\alpha > 0$, (A.9), and (A.8) implies the desired result (A.7). Therefore, (A.4) is true.

We are now in a position to prove (2.13).

Write

$$\mathrm{Adj}(I-AS) = I + A_1S + \cdots + A_{n-1}S^{n-1},$$
$$A(S) \triangleq \det(I-AS) = a_0 + a_1S + \cdots + a_nS^n$$

and set

$$M = \begin{bmatrix} 0 & B & A_1B & \dots & A_{n-1}B \\ a_0 & a_1 & a_2 & \dots & a_n \end{bmatrix},$$

where I denotes $n \times n$ identity matrix.

Clearly, we have

$$\begin{aligned} A(S)\psi_\lambda &= \begin{bmatrix} \mathrm{Adj}(I-AS)S^2Bu_\lambda + \mathrm{Adj}(I-AS)(x_0\lambda + CSw_\lambda) \\ A(S)Su_\lambda \end{bmatrix} \\ &= MSU_t + \begin{bmatrix} \mathrm{Adj}(I-AS)(x_0\lambda + CSw_\lambda) \\ 0 \end{bmatrix}, \end{aligned}$$

where

$$U_t = [u_t,\ Su_t, \dots, S^nu_t]^\tau.$$

Therefore, we get

$$\begin{aligned} &\lambda_{\min}\left(\int_{T^k}^t [A(S)\psi_s][A(S)\psi_s]^\tau ds\right) \\ \text{(A.11)} \quad &\geq \frac{1}{2}\lambda_{\min}\left(M\int_{T^k}^t (SU_s)(SU_s)^\tau ds M^\tau\right) - c\sum_{i=0}^n \int_0^t \left[s^{2i} + \|S^iw_s\|^2\right] ds. \end{aligned}$$

Using the argument in, e.g., Zhang and Chen [1991] we can obtain

$$\text{(A.12)} \qquad \int_0^t \|S^iw_s\|^2 ds \leq ct^{2i+3}, \quad i = 0,\ 1,\ 2,\ \dots,\ n,$$

and

$$\begin{aligned} &\lambda_{\min}\left(\int_{T^k}^t [A(S)\psi_s][A(S)\psi_s]^\tau ds\right) = \min_{\|x\|=1} \int_{T^k}^t \left|\sum_{i=0}^n a_iS^ix^\tau\psi_s\right|^2 ds \\ \text{(A.13)} \quad &\leq c\sum_{i=0}^n t^{2i+1}\lambda_{\min}\left(\int_{T^k}^t \psi_s\psi_s^\tau ds\right) + c\left(\sum_{i=0}^n t^{2i+1}\right)\int_0^{T^k} \|\psi_s\|^2 ds. \end{aligned}$$

From (A.11)–(A.13) we have

$$\begin{aligned} \lambda_{\min}\left(\int_{T^k}^t \psi_s\psi_s^\tau ds\right) &\geq \frac{1}{c}\lambda_{\min}(MM^\tau)\left(\sum_{i=0}^n t^{2i+1}\right)^{-1}\lambda_{\min}\left(\int_{T^k}^t (SU_s)(SU_s)^\tau ds\right) \\ \text{(A.14)} \quad &\quad -c\left(\sum_{i=0}^n t^{2i+1}\right)^{-1}\left(\sum_{i=0}^{n+1} t^{2i+1}\right) - \int_0^{T^k} \|\psi_s\|^2 ds. \end{aligned}$$

By induction we can show that

$$|S^i u_t - S_k^i u_t| \le t^{i-1} \int_0^{T^k} |u_s| ds \quad \forall i = 1,\ 2, \dots, \quad \forall t \in [T^k,\ T^{k+1}).$$

From this we see that for any $x \in \mathbb{R}^{n+1}$ with $\|x\| = 1$ and $t \in [T^k,\ T^{k+1})$,

$$\begin{aligned}
\int_{T^k}^t (x^\tau S U_s)^2 ds &\ge \frac{1}{2} \int_{T^k}^t [x^\tau S_k U_s(k)]^2 ds - \int_{T^k}^t \|S_k U_s(k) - S U_s\|^2 ds \\
&\ge \frac{1}{2} \int_{T^k}^t [x^\tau S_k U_s(k)]^2 ds - \int_{T^k}^t \sum_{i=1}^{n+1} |S^i u_s - S_k^i u_s|^2 ds \\
&\ge \frac{1}{2} \int_{T^k}^t [x^\tau S_k U_s(k)]^2 ds - \sum_{i=1}^{n+1} \left(\int_0^{T^k} |u_s| ds \right)^2 \int_{T^k}^t s^{2(i-1)} ds \\
&\ge \frac{1}{2} \int_{T^k}^t [x^\tau S U_s(k)]^2 ds - \sum_{i=1}^{n+1} \left(\int_0^{T^k} u_s^2 ds \right) T^{2i(k+1)} \\
&\ge \frac{1}{2} \int_{T^k}^t [x^\tau S U_s(k)]^2 ds - (n+1) T^{2(n+1)(k+1)} \int_0^{T^k} u_s^2 ds,
\end{aligned}$$

which implies that

$$\begin{aligned}
\lambda_{\min} \left(\int_{T^k}^{T^{k+1}} (SU_s)(SU_s)^\tau ds \right) &\ge \frac{1}{2} \lambda_{\min} \left(\int_{T^k}^{T^{k+1}} (S_k U_s(k))(S_k U_s(k))^\tau ds \right) \\
&\quad - (n+1) T^{2(n+1)(k+1)} L_{T^k}.
\end{aligned} \tag{A.15}$$

This together with (A.4) and (A.14) leads to

$$\lambda_{\min} \left(\int_{T^k}^{T^{k+1}} \psi_s \psi_s^\tau ds \right) \ge c^{-1} T^{-(2n+1)(k+1)} \gamma^{T^{k+1}} L_{T^k}^2 - c T^{(2n+3)(k+1)} L_{T^k}. \tag{A.16}$$

With $(S+1) z_t = \psi_t$ in mind, we get

$$\begin{aligned}
\lambda_{\min} \left(\int_{T^k}^{T^{k+1}} \psi_s \psi_s^\tau ds \right) &= \lambda_{\min} \left(\int_{T^k}^{T^{k+1}} [(S+1) z_s][(S+1) z_s]^\tau ds \right) \\
&= \min_{\|x\|=1} \int_{T^k}^{T^{k+1}} \left| x^\tau z_s + S x^\tau z_s \right|^2 ds \\
&\le 4 T^{2(k+1)} \lambda_{\min} \left(\int_{T^k}^{T^{k+1}} z_s z_s^\tau ds \right) + 2 T^{2(k+1)} \int_0^{T^k} \|z_s\|^2 ds,
\end{aligned}$$

i.e.,

$$\begin{aligned}
\lambda_{\min} \left(\int_{T^k}^{T^{k+1}} z_s z_s^\tau ds \right) &\ge 4^{-1} T^{-2(k+1)} \lambda_{\min} \left(\int_{T^k}^{T^{k+1}} \psi_s \psi_s^\tau ds \right) \\
&\quad - 2^{-1} \int_0^{T^k} \|z_s\|^2 ds.
\end{aligned}$$

From this, (A.16), and the definition of L_{T^k} it follows that

$$\lambda_{\min}\left(\int_{T^k}^{T^{k+1}} z_s z_s^\tau ds\right) \geq c^{-1} T^{-(2n+3)(k+1)} \gamma^{T^{k+1}} L_{T^k}^2 - c T^{(2n+1)(k+1)} L_{T^k},$$

which implies the desired result (2.13). □

Appendix B. The following lemma is based on Chen [1992].

LEMMA B.1. *If $T > 1$ is a constant, (F, C) is controllable, $F \in \mathcal{F}_\sigma$ with $\sigma < \infty$ a.s. being a stopping time, and if*

$$\limsup_{k\to\infty} \frac{1}{T^k} \int_0^{T^k} \|x_s\|^2 ds < \infty \quad a.s. \tag{B.1}$$

for the system

$$dx_t = F x_t dt + C dw_t, \quad t \geq \sigma, \tag{B.2}$$

then F must be stable a.s.

Proof. Assume that F^τ has an eigenvalue λ with $\Re(\lambda) \geq 0$, where $\Re(x)$ denotes the real part of a complex number x. Let y be the corresponding eigenvector, i.e., $F^\tau y = \lambda y$. Then by (B.2) we get

$$d[\Re(y^\tau x_t) + i\Im(y^\tau x_t)] = [\Re(\lambda) + i\Im(\lambda)][\Re(y^\tau x_t) + i\Im(y^\tau x_t)]dt + y^\tau C dw_t \quad \forall t \geq \sigma,$$

i.e.

$$dz_t = \begin{bmatrix} \Re(\lambda) & -\Im(\lambda) \\ \Im(\lambda) & \Re(\lambda) \end{bmatrix} z_t dt + \begin{bmatrix} \Re(y^\tau C) \\ \Im(y^\tau C) \end{bmatrix} dw_t, \tag{B.3}$$

where $\Im(x)$ denotes the imaginary part of a complex number x and

$$z_t = \begin{bmatrix} \Re(y^\tau x_t) \\ \Im(y^\tau x_t) \end{bmatrix}.$$

Using Ito's formula, by (B.3) we obtain

$$\begin{aligned} dz_t^\tau z_t &= 2 z_t^\tau \begin{bmatrix} \Re(\lambda) & -\Im(\lambda) \\ \Im(\lambda) & \Re(\lambda) \end{bmatrix} z_t dt + 2 z_t^\tau \begin{bmatrix} \Re(y^\tau C) \\ \Im(y^\tau C) \end{bmatrix} dw_t + \|y^\tau C\|^2 dt \\ &= 2\Re(\lambda)\|z_t\|^2 dt + 2 z_t^\tau \begin{bmatrix} \Re(y^\tau C) \\ \Im(y^\tau C) \end{bmatrix} dw_t + \|y^\tau C\|^2 dt, \end{aligned}$$

which implies (Christopeit [1986]) that for any $\eta \in (0, 1/2)$,

$$\|z_t\|^2 = \|z_\sigma\|^2 + 2\Re(\lambda) \int_\sigma^t \|z_s\|^2 ds + O\left(\left(\int_\sigma^t \|z_s\|^2 ds\right)^{\frac{1}{2}+\eta}\right) + \|y^\tau C\|^2 (t - \sigma). \tag{B.4}$$

Noting that (B.1) implies that

$$\limsup_{k\to\infty} \frac{1}{T^k} \int_\sigma^{T^k} \|z_s\|^2 ds < \infty, \tag{B.5}$$

by (B.4) we have

$$\text{(B.6)}\quad \|z_t\|^2 = \|z_\sigma\|^2 + 2\Re(\lambda)\int_\sigma^t \|z_s\|^2 ds + O\left(t^{\frac{1}{2}+\eta}\right) + \|y^\tau C\|^2(t-\sigma) \quad \forall\eta\in(0,1/2).$$

From controllability of (F, C) it is easy to see that $\|y^\tau C\| \neq 0$, and hence from (B.6) and $\Re(\lambda) \geq 0$ it follows that for some $t_0 \geq \sigma$ and $c > 0$,

$$\|z_t\|^2 \geq ct \quad \forall t \geq t_0, \quad t_0 \text{ random},$$

which contradicts (B.5). □

Appendix C. Proof of Lemma 3.3. As in Appendix A, we can show that there exist constants $c > 0$, $\gamma > 1$, and k_0 such that

$$\text{(C.1)}\qquad \lambda_{\min}\left(\int_{T^k}^{T^{k+1}} (S_k U_s(k))(S_k U_s(k))^\tau ds\right) \geq c\gamma^{T^{k+1}} L_{T^k}^2 \quad \forall k \geq k_0,$$

where here and hereafter

$$U_t(k) = [u_t,\ S_k u_t, \ldots, S_k^{p+q-1} u_t]^\tau.$$

Set $W_t = y_0 t + SC(S)w_t + S^2\eta_t$ and $M = [M_1,\ M_2]^\tau$ with

$$M_1^\tau \triangleq \left.\overbrace{\begin{pmatrix} 0 & b_1 & \cdots & \cdots & \cdots & \cdots & b_q & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & & \ddots & 0 \\ 0 & \cdots & 0 & 0 & b_1 & \cdots & \cdots & \cdots & \cdots & b_q \end{pmatrix}}^{p+q}\right\} p$$

and

$$M_2^\tau \triangleq \left.\overbrace{\begin{pmatrix} 1 & a_1 & \cdots & \cdots & \cdots & \cdots & a_q & 0 & \cdots & 0 \\ 0 & 1 & \ddots & & & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & & \ddots & 0 \\ 0 & \cdots & 0 & 1 & a_1 & \cdots & \cdots & \cdots & \cdots & a_q \end{pmatrix}}^{p+q}\right\} q.$$

Then from (3.1) and (3.11) it follows that

$$\text{(C.2)}\qquad A(S)\varphi_t = MSU_t + [W_t,\ SW_t, \ldots, S^{p-1}W_t,\ \underbrace{0, \ldots, 0}_{q}]^\tau,$$

where here and hereafter,

$$\text{(C.3)}\qquad U_t = [u_t,\ Su_t, \ldots, S^{p+q-1}u_t]^\tau.$$

Notice that by Assumption A.2, for $i = 1, 2, \ldots, p$,

$$\int_0^t (S^i\eta_s)^2 ds \leq \int_0^t \frac{t^{2i} - s^{2i}}{2i(2i-1)[(i-1)!]^2}\eta_s^2 ds \leq ct^{2i+1}.$$

Then similar to (A.11)–(A.14), by (C.2) we obtain

$$\lambda_{\min}\left(\int_{T^k}^{t}\varphi_s\varphi_s^\tau ds\right) \geq \frac{1}{c}\left(\sum_{i=0}^{p} t^{2i+1}\right)^{-1}\lambda_{\min}\left(\int_{T^k}^{t}(SU_s)(SU_s)^\tau ds\right)$$

$$-c\left(\sum_{i=0}^{p} t^{2i+1}\right)^{-1}\sum_{i=0}^{p+l}(t+1)^{2i+1} - \int_0^{T^k}\|\varphi_s\|^2 ds. \tag{C.4}$$

By (C.3) and along the argument of (A.15) we get

$$\lambda_{\min}\left(\int_{T^k}^{T^{k+1}}(SU_s)(SU_s)^\tau ds\right) \geq \frac{1}{2}\lambda_{\min}\left(\int_{T^k}^{T^{k+1}}(S_kU_s(k))(S_kU_s(k))^\tau ds\right)$$

$$-(p+q)T^{2(p+q)(k+1)}L_{T^k}.$$

This together with (C.1) and (C.4) leads to

$$\lambda_{\min}\left(\int_{T^k}^{T^{k+1}}\varphi_s\varphi_s^\tau ds\right) \geq c^{-1}T^{-(2p+1)(k+1)}\gamma^{T^{k+1}}L_{T^k}^2 - cT^{2(p+q+l+2)(k+1)}L_{T^k}. \tag{C.5}$$

With $F(S)\varphi_t^f = \varphi_t$ in mind, as in (A.13) we get

$$\lambda_{\min}\left(\int_{T^k}^{T^{k+1}}\varphi_s\varphi_s^\tau ds\right) = \lambda_{\min}\left(\int_{T^k}^{T^{k+1}}(F(S)\varphi_s^f)(F(S)\varphi_s^f)^\tau ds\right)$$

$$= \min_{\|x\|=1}\int_{T^k}^{T^{k+1}}\left|\sum_{i=0}^{l+1} f_i S^i x^\tau \varphi_s^f\right|^2 ds$$

$$\leq cT^{(2l+3)(k+1)}\lambda_{\min}\left(\int_{T^k}^{T^{k+1}}\varphi_s^f(\varphi_s^f)^\tau ds\right) + cT^{(2l+3)(k+1)}\int_0^{T^k}\|\varphi_s^f\|^2 ds,$$

i.e.,

$$\lambda_{\min}\left(\int_{T^k}^{T^{k+1}}\varphi_s^f(\varphi_s^f)^\tau ds\right) \geq cT^{-(2l+3)(k+1)}\lambda_{\min}\left(\int_{T^k}^{T^{k+1}}\varphi_s\varphi_s^\tau ds\right)$$

$$-\int_0^{T^k}\|\varphi_s^f\|^2 ds.$$

From this, (C.5) and the definition of L_{T^k} it follows that

$$\lambda_{\min}\left(\int_{T^k}^{T^{k+1}}\varphi_s^f(\varphi_s^f)^\tau ds\right) \geq c^{-1}T^{-2(p+l+2)(k+1)}\gamma^{T^{k+1}}L_{T^k}^2 - cT^{2(p+q+l+2)(k+1)}L_{T^k},$$

which implies the desired result, Lemma 3.3. □

Appendix D.

LEMMA D.1. *If $A(S)$ and $SB(S)$ are coprime, $b_q \neq 0$ and*

$$\theta_{T^k} \xrightarrow[k\to\infty]{} \theta \ a.s.,$$

then there is an integer-valued K, possibly depending on a sample path such that for all $k \geq K$, $A_k(S)G_k(S) - SB_k(S)H_k(S) = E(S)$ is solvable with respect to $G_k(S)$ and $H_k(S)$ subject to $\partial(G_k(S)) \leq q-1$ and $\partial(H_k(S)) = p$, and such that $\|G_k(S)\|^2 + \|H_k(S)\|^2 \leq k/(2C_e(p+q+1))$.

Proof. Let

$$
\text{(D.1)}\quad M_3^\tau = \left.\overbrace{\begin{pmatrix} 1 & a_1 & \cdots & \cdots & \cdots & \cdots & a_p & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & & & & & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & a_1 & \cdots & \cdots & \cdots & \cdots & a_p & 0 \end{pmatrix}}^{p+q+1}\right\} q,
$$

$$
\text{(D.2)}\quad M_4^\tau = \left.\overbrace{\begin{pmatrix} 0 & -b_1 & \cdots & \cdots & \cdots & \cdots & -b_q & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & & & \ddots & 0 \\ 0 & \cdots & 0 & 0 & -b_1 & \cdots & \cdots & \cdots & \cdots & -b_q \end{pmatrix}}^{p+q+1}\right\} p+1,
$$

$$
\text{(D.3)}\quad M' = [M_3,\ M_4], \qquad H_e = [1,\ e_1,\ \ldots,\ e_{p+q}]^\tau.
$$

Replacing a_i and b_j by their estimates a_{iT^k} and b_{jT^k}, respectively, in (D.1)–(D.3) for $i = 1, \ldots, p$ and $j = 1, \ldots, q$, we correspondingly denote M_3, M_4, and M' by $M_{3,k}$, $M_{4,k}$, and M'_k. Furthermore, if M'_k is nonsingular, we set $\Psi_k = (M'_k)^{-1}H_e$.

Since $A(S)$ and $SB(S)$ are coprime and $b_q \neq 0$, we see that M' given by (D.1)–(D.3) is nonsingular. Let $\Psi = (M')^{-1}H_e$ and $G(S) = \sum_{i=0}^{q-1} g_i S^i$, $H(S) = \sum_{i=0}^{p} h_i S^i$ with

$$
[g_0,\ g_1, \ldots, g_{q-1},\quad h_0,\ h_1, \ldots, h_p] \triangleq \Psi.
$$

Then recalling that

$$
\theta_{T^k} \xrightarrow[k \to \infty]{} \theta,
$$

we see that there is an integer $K' \geq 0$ such that for all $k \geq K'$, M'_k is nonsingular,

$$
M'_k \xrightarrow[k \to \infty]{} M', \text{ and } \Psi_k \xrightarrow[k \to \infty]{} \Psi.
$$

Furthermore for all $k \geq K'$, if we set $G_k(S) = \sum_{i=0}^{q-1} g_{i,k} S^i$, $H_k(S) = \sum_{i=0}^{p} h_{i,k} S^i$ with

$$
[g_{0,k},\ g_{1,k}, \ldots, g_{q-1,k},\quad h_{0,k},\ h_{1,k}, \ldots, h_{p,k}] \triangleq \Psi_k,
$$

then we have $\partial(G_k(S)) \leq q-1$, $\partial(H_k(S)) = p$ and

$$
A_k(S)G_k(S) - SB_k(S)H_k(S) = E(S).
$$

Noting that

$$
\Psi_k \xrightarrow[k \to \infty]{} \Psi
$$

we see that there exists an integer $K \geq K'$ such that for all $k \geq K$, $\|G_k(S)\|^2 + \|H_k(S)\|^2 = \|\Psi_k\|^2 \leq k/(2C_e(p+q+1))$. $\square$

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